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ON THE STATIONARY MOTIONS IN A NEWTONIAN FIELD OF FORCE OF A BODY THAT ADMITS OF REGULAR POLYHEDRON SYMMETRY GROUPS*

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The author's results /1-3/ on the stationary motions in a central Newtonian field of force when the centre of mass is assumed fixed, of a fixed system of particles of equal mass, located at the vertices of a regular polyhedron, are written in a general mathematical form and are extended to any fixed system whose mass distribution admits of one of the symmetry groups of a regular polyhedron (tetrahedron, octahedron, or icosahedron). It is shown that the results obtained earlier by considering the first terms of the Taylor expansion of the force function are preserved when account is taken of the full expression for the force function (potential). The stability of these solutions is investigated.

1. We consider the motion of a rigid body with a fixed point in a central Newtonian field of force. Let the fixed point G be at the centre of mass, and let the mass distribution in the body be invariant under transformations that belong to one of the discrete symmetry groups: the tetrahedron, octahedron, or icosahedron.

Let $O\xi\eta\zeta$ be a fixed coordinate system with origin at the attracting centre O, and $Gxyz$ a dimensionless coordinate system rigidly connected with the body (the scale of length is a characteristic dimension l of the body). The force function of Newtonian gravitation

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is

$$U(\gamma)=\frac{fM_0}{R}\iiint\frac{l^3\,dm}{\sqrt{1+2\varepsilon(\gamma\cdot\tau)+\varepsilon^2\tau^2}} \tag{1.1}$$

$$\gamma=(\gamma_1,\gamma_2,\gamma_3)=\mathbf{R}/R,\quad \varepsilon=l/R,\quad \tau=\mathbf{r}/l,\quad \mathbf{R}=\mathbf{OG},\quad \mathbf{r}=\mathbf{GA}$$

where A is an arbitrary pont of the body, f is the gravitational constant, and M_0 is the mass of the attracting body. Throughout, the integration is performed over the volume of the body.

Assertion. Let S be a transformation of the symmetry group, allowed by the mass distribution in the body. Then, $U(\gamma)=U(S^T\gamma)$.

Proof. By (1.1),

$$U(\gamma)=\frac{fM_0}{R}\iiint\frac{l^3\,dm}{\sqrt{1+2\varepsilon(\gamma\cdot\tau)+\varepsilon^2\tau^2}}=$$

$$\frac{fM_0}{R}\iiint\frac{l^3\,dm}{\sqrt{1+2\varepsilon(\gamma\cdot S\tau)+\varepsilon^2(S\tau)^2}}=U(S^T\gamma)$$

where $\det S^T=1$, $(S\tau)^2=\tau^2$, since the matrix S is orthogonal.

On expanding the integrand in powers of ε, we can write Eq.(1.1) as

$$U(\gamma)=\sum_{n=0}^{\infty}Q_n(\gamma_1,\gamma_2,\gamma_3)\,\varepsilon^n$$

where $Q_n(\gamma)$ are polynomials in the components of the vector γ, invariant under the action of the symmetry group: $Q_n(\gamma)=Q_n(S\gamma)$. By the theorem on the polynomial invariants of discrete groups /4/, the coefficients $Q_n(\gamma)$ can be written as $Q_n(\gamma)=P_n(I_1,I_2,I_3)$, where $I_1(\gamma)$, $I_2(\gamma)$, $I_3(\gamma)$ are polynomial invariants of the group, and P_n are polynomials in I_1, I_2, I_3. The expressions for these invariants in terms of $\gamma_1,\gamma_2,\gamma_3$ are as follows /5, 6/: $I_1=\gamma_1^2+\gamma_2^2+\gamma_3^2$

for the tetrahedron

$$I_2=\gamma_1\gamma_2\gamma_3,\quad I_3=\gamma_1^2\gamma_2^2+\gamma_2^2\gamma_3^2+\gamma_3^2\gamma_1^2$$

for the octahedron and cube

$$I_2=\gamma_1^2\gamma_2^2+\gamma_2^2\gamma_3^2+\gamma_3^2\gamma_1^2,\quad I_3=\gamma_1^2\gamma_2^2+\gamma_3^2$$

for the dodecahedron and icosahedron

$$I_2=4(\gamma_1^2+\gamma_2^2+\gamma_3^2)+3(5-\sqrt{5})(\gamma_1^4\gamma_2^2+\gamma_2^4\gamma_3^2+\gamma_3^4\gamma_1^2)+$$
$$3(5+\sqrt{5})(\gamma_1^2\gamma_2^4+\gamma_2^2\gamma_3^4+\gamma_3^2\gamma_1^4)$$
$$I_3=5(\gamma_1^{10}+\gamma_2^{10}+\gamma_3^{10})+9(5-2\sqrt{5})(\gamma_1^8\gamma_2^2+\gamma_2^8\gamma_3^2+$$
$$\gamma_3^8\gamma_1^2)+9(5+2\sqrt{5})(\gamma_1^2\gamma_2^8+\gamma_2^2\gamma_3^8+\gamma_3^2\gamma_1^8)+21(5-\sqrt{5})(\gamma_1^6\gamma_2^4+$$
$$\gamma_2^6\gamma_3^4+\gamma_3^6\gamma_1^4)+21(5+\sqrt{5})(\gamma_1^4\gamma_2^6+\gamma_2^4\gamma_3^6+\gamma_3^4\gamma_1^6)$$

The force function is therefore

$$U_\nu=U_\nu(\varepsilon,I_1,I_2,I_3)$$

The values $\nu=1,2,3$, correspond to the tetrahedron, octahedron, and icosahedron groups respectively.

Since the field of force is axisymmetric and the force function depends only on $\gamma_1,\gamma_2,\gamma_3$, all the equilibrium positions of the rigid body are independent of the rotation about γ.

The equations of motion of the body can be written as

$$\frac{d(J\omega)}{dt}=\frac{\partial U_\nu}{\partial\gamma}\times\gamma,\quad \frac{d\gamma}{dt}=\gamma\times\omega \tag{1.2}$$

(J is the moment of inertia, and $\omega=(\omega_1,\omega_2,\omega_3)$ is the angular velocity vector). For the present bodies, the central tensor of inertia of second order has spherical symmetry.

The equations of motion (1.2) admit of the energy integral, area integral and geometric integral

$$H_\nu={}^1\!/_2 J(\omega_1^2+\omega_2^2+\omega_3^2)-U_\nu=h=\text{const}$$
$$K=J(\omega_1\gamma_1+\omega_2\gamma_2+\omega_3\gamma_3)=k=\text{const},\quad I_1=\gamma_1^2+\gamma_2^2+\gamma_3^2=1$$

By Routh's theorem /7/ the problem of stationary motions amounts to finding the stationary values of the function

$$W_\nu = {}^1/_2 J(\omega_1^2 + \omega_2^2 + \omega_3^2) - U_\nu(\varepsilon, 1, I_2, I_3) - \lambda(K - k) + {}^1/_2\mu(I_1 - 1)$$

where λ and μ are Lagrange undetermined multipliers. The conditions for the function to be stationary are given by a system of equations which admits of the following one-parameter $(\omega_i = \lambda\gamma_i)$ families of solutions:

for the tetrahedron

$$\gamma_1 = 0,\ \gamma_2 = 0,\ \gamma_3 = \pm 1 \quad (1\ 2\ 3) \tag{1.3}$$

$$\gamma_1 = \pm 1/\sqrt{3},\ \gamma_2 = \pm 1/\sqrt{3},\ \gamma_3 = \pm 1/\sqrt{3} \tag{1.4}$$

for the cube and octahedron (1.3), (1.4) and

$$\gamma_1 = 0, \quad \gamma_2 = \pm 1/\sqrt{2}, \quad \gamma_3 = \pm 1/\sqrt{2} \quad (1\,2\,3) \tag{1.5}$$

for the dodecahedron and icosahedron (1.3) and

$$\gamma_1 = 0, \quad \gamma_2 = \pm\sqrt{\frac{5+\sqrt{5}}{10}}, \quad \gamma_3 = \pm\sqrt{\frac{5-\sqrt{5}}{10}} \quad (1\,2\,3) \tag{1.6}$$

$$\gamma_1 = 0, \quad \gamma_2 = \pm\sqrt{\frac{3-\sqrt{5}}{6}}, \quad \gamma_3 = \pm\sqrt{\frac{3+\sqrt{5}}{6}} \quad (1\,2\,3) \tag{1.7}$$

$$\mu = J\lambda^2 + 3(\gamma_1\gamma_2\gamma_3)U_{\nu,2} + 4(\gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2)U_{\nu,3}$$

(where (1 2 3) denotes circular permutation of the subscripts 1, 2, 3).

These solutions are the same (up to 45° rotation of the coordinate system $Gxyz$ about the x axis) as those found earlier in /2, 3/.

2. We shall study the stability of these stationary motions with respect to the quantities $\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3$. For this, we calculate the second variation $\delta^2 W_\nu$ $(\nu = 1, 2, 3)$ in the linear manifold $\delta K = 0$, $\delta I_1 = 0$. We have

$$\delta^2 W_\nu = B + (\mu - J\lambda^2)\sum_{i=1}^{3}(\delta\gamma_i)^2 - \sum_{i,j=1}^{3}\Big(U_{\nu,22}\frac{\partial I_2}{\partial\gamma_i}\frac{\partial I_2}{\partial\gamma_j} + U_{\nu,23}\frac{\partial I_2}{\partial\gamma_i}\frac{\partial I_3}{\partial\gamma_j} + \tag{2.1}$$
$$U_{\nu,33}\frac{\partial I_3}{\partial\gamma_i}\frac{\partial I_3}{\partial\gamma_j} + U_{\nu,2}\frac{\partial^2 I_2}{\partial\gamma_i\partial\gamma_j} + U_{\nu,32}\frac{\partial I_3}{\partial\gamma_i}\frac{\partial I_2}{\partial\gamma_j} + U_{\nu,3}\frac{\partial^2 I_3}{\partial\gamma_i\partial\gamma_j}\Big)(\delta\gamma_i)(\delta\gamma_j)$$
$$B = J\sum_{i=1}^{3}\Omega_i^2, \quad \Omega_i = \delta\omega_i - \lambda\delta\gamma_i, \quad U_{\nu,i} = \partial U_\nu/\partial I_i$$
$$U_{\nu,ij} = \partial^2 U_\nu/\partial I_i\partial I_j$$

For the tetrahedron, the form (2.1) on the solution (1.3) is

$$\delta^2 W_1 = B - 2\{U_{1,2}(\delta\gamma_1)(\delta\gamma_2) + U_{1,3}[(\delta\gamma_1)^2 + (\delta\gamma_2)^2]\} = B + \tag{2.2}$$
$$a_1(\delta\alpha_1)^2 + b_1(\delta\alpha_2)^2$$
$$\delta\gamma_1 = \delta\alpha_1 + \delta\alpha_2,\ \delta\gamma_2 = \delta\alpha_1 - \delta\alpha_2, \quad a_1 = -2(U_{1,2} + 2U_{1,3}),$$
$$b_1 = 2(U_{1,2} - 2U_{1,3})$$

If $a_1 > 0,\ b_1 > 0$, the degree of instability $\chi = 0$ and the motion is stable. If $a_1 > 0,\ b_1 < 0$ or $a_1 < 0,\ b_1 > 0$, the degree of instability $\chi = 1$ and the motion is unstable. Finally, if $a_1 < 0,\ b_1 < 0$, the degree of instability $\chi = 2$, in which case Routh's theorem and its converse do not enable us to conclude whether the motion is stable or unstable.

For the solution (1.4)

$$\delta^2 W_1 = C[(\delta\gamma_1)^2 + (\delta\gamma_1)(\delta\gamma_2) + (\delta\gamma_2)^2] = C[3(\delta\alpha_1)^2 + (\delta\alpha_2)^2]$$
$$C = {}^4/_3[\sqrt{3}U_{1,2} + 2U_{1,3}]$$

If $c > 0$, the degree of instability $\chi = 0$ and the motion is stable, while if $c < 0$, then $\chi = 2$.

For the cube and octahedron $(\nu = 2)$, or the icosahedron and dodecahedron $(\nu = 3)$, the form (2.1) is

$$\delta^2 W_v = B + C_v (\delta\gamma_1)^2 + d_v (\delta\gamma_2)^2 \tag{2.3}$$
$$\delta^2 W_2 = B + a_2 (\delta\alpha_1)^2 + b_2 (\delta\alpha_2)^2 \tag{2.4}$$

and the conclusions about stability or instability are just the same as above, depending on the signs of C_v, d_v or a_2, b_2.

For the solutions (1.3) we have the form (2.3) with

$$C_2 = d_2 = -2U_{2,2}, \quad C_3 = 2l^-, \quad d_3 = 2l^+$$
$$l^\pm = 3(\sqrt{5} \pm 1)U_{3,2} + 2(9\sqrt{5} \pm 10)U_{3,3}$$

For the solutions (1.4) we have the form (2.4) with

$$a_2 = b_2 = {}^8/_9\, (3U_{2,2} + U_{2,3})$$

for the solutions (1.5) we have the form (2.3)

$$C_2 = -(U_{2,2} + {}^1/_2 U_{2,3}), \quad d_2 = 4U_{2,2}$$

For the solutions (1.6)

$$C_3 = -{}^4/_{25}\,(605 - 252\sqrt{5})U_{3,3}, \quad d_3 = -{}^2/_{25}\,\{9(5 + \sqrt{5})[4U_{3,22} + 48U_{3,23} + 144U_{3,33} + 35U_{3,2}]\} + 2(3064 + 487\sqrt{5})U_{3,3}$$

for the solutions (1.7)

$$C_3 = \frac{4}{9}[102U_{3,2} + (155 - 28\sqrt{5})U_{3,3}], \quad d_3 = \frac{1}{2}\Big\{10(3 - \sqrt{5}\Big[27U_{3,22} + 340U_{3,23} - \frac{38900}{27}U_{3,33} + \frac{243}{5}U_{3,2}\Big] - 9(447\sqrt{5} - 935)U_{3,3}\Big\}$$

3. Taking the example of the tetrahedron, we consider whether there exist in general other stationary motions apart from (1.3), (1.4). By the results of Sect. 1, the varied potential energy is $U_1 = U_1(\varepsilon, I_1, I_2, I_3)$.

We make a change of variables, and instead of $\gamma_1, \gamma_2, \gamma_3$ take I_1, I_2, I_3. This change is well defined in the domain where the Jacobian $|\partial I/\partial\gamma| \neq 0$, i.e., outside the set

$$M = \{\gamma_1, \gamma_2, \gamma_3;\ (\gamma_1^2 - \gamma_2^2)(\gamma_2^2 - \gamma_3^2)(\gamma_3^2 - \gamma_1^2) = 0\}$$

which is a set of planes. The invariant I_1 is the same as the geometric integral and is equal to 1. The stationary motions outside the set M are thus given by the relations

$$U_{1,2}' = 0, \ U_{1,3}' = 0; \ U_1' = (\varepsilon, 1, I_2, I_3) \tag{3.1}$$

The solution of Eqs.(3.1) depends on the actual mass distribution in the body. We shall not treat in general the auxiliary conditions under which such solutions exist. Note that all the above solutions (1.3), (1.4) lie in the set M.

Consider one of the planes that form the set M, say $M_{12} = \{\gamma_1, \gamma_2, \gamma_3;\ \gamma_1 = \gamma_2\}$. Then, in the set $M_{12} \cap \{I_1 = 1\}$, the invariants I_2, I_3 can be written as

$$I_2 = f(\gamma_3) = {}^1/_2\,(1 - \gamma_3^2)\,\gamma_3, \quad I_3 = g(\gamma_3) = {}^1/_4\,(-3\gamma_3^4 + 2\gamma_3^2 + 1)$$

Then, in the set

$$M_{12} \setminus \{\gamma_1 = \gamma_2 = 0\} \tag{3.2}$$

the necessary condition for the existence of a conditional extremum of the varied potential energy W_1 leads to the equations

$$\partial U''/\partial\gamma_3 = (1 - 3\gamma_3^2)({}^1/_2 U_{1,2}'' + \gamma_3 U_{1,3}'') = 0$$
$$U_1''(\gamma_3) = U_1[\varepsilon, 1, f(\gamma_3), g(\gamma_3)] \tag{3.3}$$

Equating the first factor in (3.3) to zero, we obtain our above solution (1.4). The conditions for the existence of other solutions of (3.3), which involve the vanishing of the second factor, depend on the actual mass distribution and will not be considered here. Finally, $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = \pm 1$ is also a solution of the equations of equilibrium (1.3), as may easily be verified directly.

To sum up, apart from (1.3) and (1.4), there are no other general solutions which are independent of the mass distribution and which withstand a transformation of the tetrahedron symmetry group.

Consider an example. In /2/ we studied the motion in a Newtonian field of force of a tetrahedron, at whose vertices are located particles of equal mass, under the assumption that the tetrahedron centre of mass is the same as the fixed point. The potential was written as a series in powers of ε, of the form

$$U_1 = A_0 + A_1\varepsilon + A_2\varepsilon^2 + A_3 I_2 \varepsilon^3 + A_4 I_3 \varepsilon^4 + \ldots$$

In this case Eqs.(3.1) take the form $A_3\varepsilon^3 + \ldots = 0$, $A_4\varepsilon^4 + \ldots = 0$, and for sufficiently small $\varepsilon \neq 0$, have no solutions. Hence the only solutions are in the planes that form the set M.

Consider, say, the plane M_{12}. On the set $M_{12} \cap \{I_1 = 1\}$ the potential is

$$U_1' = A_0 + A_1\varepsilon + A_2\varepsilon^2 + A_3\varepsilon^3 f(\gamma_3) + A_4\varepsilon^4 g(\gamma_3) + \ldots$$

so that, on the set (3.2), we can write Eq.(3.3) as $A_3 \cdot {}^1/_2 (1 - 3\varepsilon_3{}^2)\varepsilon^3 + \ldots = 0$, i.e., for sufficiently small $\varepsilon \neq 0$, there are no other solutions in the plane apart from (1.3) and (1.4).

By considering the other planes of the set M, we can show that, for sufficiently small $\varepsilon \neq 0$, the problem stated in /2/ has no solutions other than (1.3), (1.4).

Similar arguments hold for other bodies which have the symmetries of regular polyhedra.

4. All our results also hold for bodies of this type in a circular orbit, since, due to the equality of all the principal moments of inertia of the body, there are no moments of central and Coriolis forces in the orbital coordinate system.

For the above problems, when there are no stationary motions apart from (1.3) and (1.4), the bifurcation diagram in the plane of constant energy integral h and area integral k consists of three parabolas. These parabolas form a set, in which readjustment occurs of the domains of possible motion /8/, given by the relation $-U_\nu \leqslant h$.

If there are other stationary motions in the problem, extra branches appear on the bifurcation diagram.

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